Radiation from a Uniformly Moving Charge in an Anisotropic Plasma*

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Summary—**The radiation from a point charge moving uniformly in a plasma is investigated when the charge is moving in the direction of an external magnetic field. In general there are two modes, for each of which all the components of the electric and magnetic field are present. The two parameters of interest in this problem are the ratio u/c_0 of the velocity of the charges to the free-space velocity of electromagnetic waves and the ratio R of the gyromagnetic frequency to the plasma frequency of the electrons. For two sets of values of these parameters the frequency and the angular spectrum of the emitted radiation are obtained. In certain cases, as many as three Cerenkov rays are found to propagate in the same direction; these multiple rays, however, correspond to different frequency components and to different modes of propagation. The motivation for this investigation is indicated briefly.**

INTRODUCTION

A LOW-FREQUENCY radiation has been observed from space vehicles in their passage through ionized regions in interplanetary space. The investigation of possible sources of this radiation is of current interest. It is possible for charges that collect on the surface of the space vehicle to radiate in the course of their motion. Such radiation can have a low frequency. In order to understand the nature of the latter type of radiation, the problem of radiation from a point charge moving uniformly in an anisotropic plasma is examined in detail in this paper.

The plasma is assumed to be incompressible; it is idealized so that it has the properties of a dielectric. A uniform magnetic field is assumed to be maintained throughout the plasma. The radiation characteristics of a charge moving uniformly in the direction of the external magnetic field are investigated. It is found that, in general, the radiation consists of two modes. In all the cases for which numerical computation of the frequency spectrum has been carried out, the radiation was found to be always in the lower end of the frequency spectrum. In the limiting case of an infinite external magnetic field, the radiation consists of all frequencies lower than the plasma frequency. This radiation is of the Cerenkov type and its angular spectrum has been evaluated for two sets of values of parameters of interest. In certain cases, as many as three Cerenkov rays are found to propagate in the same direction. These multiple Cerenkov rays correspond to different frequency components and to different modes of propagation.

The theoretical interpretation of Cerenkov radiation in an isotropic dielectric was first given by Frank and Tamm.[1] The Cerenkov radiation in anisotropic media such as uniaxial ferrite crystals, has been investigated, among others, by Pafmov[2,3] and Ginzburg.[4] Kolomenskii[5] and Sitenko and Kolomenskii[6] have examined certain aspects of the problem of radiation by a uniformly moving charged particle in a plasma with an external magnetic field. In their papers Kolomenskii and Sitenko have not systematically investigated the frequency spectrum of the ordinary and the extraordinary rays but have given only the frequency limits within which the two kinds of rays propagate. Moreover, the relative strengths of the Cerenkov rays of different frequency components and their angular distribution have not been given.

An excellent treatment of the general field of Cerenkov radiation together with its applications may be found in the book by Jelley[7] and the review article by Bolotovskii.[8]

More recently, Majumdar[9] has treated the radiation by a charged particle passing through an electron plasma in an external magnetic field taking into account the compressibility of the medium. There have been a number of investigations of a somewhat practical nature on Cerenkov radiation from a bunched beam in a bounded medium such as, for example, those by Coleman[10] and Kenyon.[11]

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Formulation of the Problem

Consider an unbounded, homogeneous plasma which, for the sake of simplicity, is idealized to be an incompressible, loss-free electron fluid, with stationary ions that neutralize the electrons on the average. An external magnetic field is assumed to be uniformly impressed throughout the plasma in the y direction; ϕ, ρ and y form a cylindrical coordinate system (Fig. 1). Let

$$q = q_o \frac{\delta(y - ut)\delta(\rho)}{2\pi\rho} \tag{1}$$

represent a point charge q_o moving with a uniform velocity u along the y axis. The current density arising from this uniformly moving charge is given by

$$J(r, t) = \hat{y} q_o u \frac{\delta(y - ut)\delta(\rho)}{2\pi\rho}, \tag{2}$$

where r represents the position vector of a point in the ϕ, ρ, y space. It is assumed that the source (2) is sufficiently weak so that linearized plasma theory is applicable.

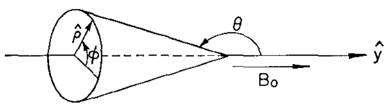

Fig. 1—Geometry of the problem.

Let $E(r, t)$ and $H(r, t)$ be, respectively, the electric and magnetic field vectors. It is convenient to apply the Fourier transform pair

$$f(r, \omega) = \int_{-\infty}^{\infty} f(r, t) e^{i\omega t} dt \tag{3}$$

$$f(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(r, \omega) e^{-i\omega t} d\omega \tag{4}$$

to the source (2) and the field quantities. The Fourier transform of the source (2) is obtained as

$$J_y(r, \omega) = q_o \frac{\delta(\rho)}{2\pi\rho} e^{i(\omega/u)y}. \tag{5}$$

The electric and magnetic fields $E(r, \omega)$ and $H(r, \omega)$ are known to satisfy, in the frequency domain, the following Maxwell's equations:

$$\nabla \times E(r, \omega) = i\omega\mu_o H(r, \omega) \tag{6}$$

$$\nabla \times H(r, \omega) = -i\omega\epsilon_o \boldsymbol{\varepsilon} \cdot E(r, \omega) + \hat{y} J_y(r, \omega). \tag{7}$$

The components of the relative dyadic dielectric constant $\boldsymbol{\varepsilon}$ are given by the following matrix:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \epsilon_1 & i\epsilon_2 & 0 \\ -i\epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix}, \tag{8}$$

where

$$\epsilon_1 = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1}$$

$$\epsilon_2 = \left(\frac{\omega_p}{\omega}\right)^2 \left[\frac{\omega}{\omega_c} - \frac{\omega_c}{\omega}\right]^{-1}$$

and

$$\epsilon_3 = 1 - \left(\frac{\omega_p}{\omega}\right)^2. \tag{9}$$

In (9) ω_p and ω_c are, respectively, the plasma and the gyromagnetic frequencies.

It is obvious that the field components are independent of ϕ and are dependent on y only through the phase factor $e^{i\omega y/u}$. It is convenient to separate out the y dependence as follows:

$$E(r, \omega) = E(\rho, \omega) e^{i(\omega/u)y}$$

$$H(r, \omega) = H(\rho, \omega) e^{i(\omega/u)y}$$

$$J_y(r, \omega) = J_y(\rho, \omega) e^{i(\omega/u)y}. \tag{10}$$

With the help of (6), (7) and (10) it is possible to express $H_\rho(\rho, \omega)$, $H_y(\rho, \omega)$, $E_\rho(\rho, \omega)$ and $E_y(\rho, \omega)$ in terms of $E_\phi(\rho, \omega)$ and $H_\phi(\rho, \omega)$ as follows:

$$H_\rho(\rho, \omega) = \frac{1}{u\mu_o} E_\phi(\rho, \omega) \tag{11}$$

$$H_y(\rho, \omega) = \frac{i}{\omega\mu_o} \frac{1}{\rho} \frac{\partial}{\partial\rho} [\rho E_\phi(\rho, \omega)] \tag{12}$$

$$E_\rho(\rho, \omega) = i \frac{\epsilon_2}{\epsilon_1} E_\phi(\rho, \omega) - \frac{1}{u\epsilon_o\epsilon_1} H_\phi(\rho, \omega) \tag{13}$$

$$E_y(\rho, \omega) = \frac{1}{i\omega\epsilon_o\epsilon_3} \frac{1}{\rho} \frac{\partial}{\partial\rho} [\rho H_\phi(\rho, \omega)] + \frac{1}{i\omega\epsilon_o\epsilon_3} J_y(\rho, \omega). \tag{14}$$

Also $E_\phi(\rho, \omega)$ and $H_\phi(\rho, \omega)$ may be shown to satisfy the following coupled equations:

$$\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi(\rho, \omega))\right] + \left[\frac{\omega^2}{c_o^2}\epsilon_3 - \frac{\omega^2}{u^2}\frac{\epsilon_3}{\epsilon_1}\right] H_\phi(\rho, \omega) + \frac{i\omega^2\epsilon_o\epsilon_2\epsilon_3}{u\epsilon_1} E_\phi(\rho, \omega) = \frac{-\partial}{\partial\rho} J_y(\rho, \omega) \tag{15}$$

$$\frac{\partial}{\partial\rho}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi(\rho, \omega))\right] + \left[\frac{\omega^2}{c_o^2}\frac{\epsilon}{\epsilon_1} - \frac{\omega^2}{u^2}\right] E_\phi(\rho, \omega) - \frac{i\omega^2\mu_0\epsilon_2}{u\epsilon_1} H_\phi(\rho, \omega) = 0, \tag{16}$$

where

$$c_o^2 = \frac{1}{\mu_o\epsilon_o}; \qquad \epsilon = \epsilon_1^2 - \epsilon_2^2. \tag{17}$$

Coupled equations similar to (15) and (16) have been derived before, among others, by Kales[12] in connection with wave propagation in a ferrite medium.

Special Cases

The wave equations (15) and (16) become decoupled when 1) the external magnetic field is absent and 2) the external magnetic field is infinite. From (9) it is easily seen that $\epsilon_2=0$ in both cases and hence (15) and (16) reduce to separate wave equations for $H_\phi(\rho, \omega)$ and $E_\phi(\rho, \omega)$, respectively. The absence of a source term in (16) together with (11) and (12) leads to the result

$$E_\phi(\rho, \omega) = H_\rho(\rho, \omega) = H_y(\rho, \omega) = 0. \tag{18}$$

From (5) and (15) it follows that

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + k^2 - \frac{1}{\rho^2}\right] H_\phi(\rho, \omega) = -\frac{\partial}{\partial\rho}\left[\frac{q_o\delta(\rho)}{2\pi\rho}\right], \tag{19}$$

where

$$k^2 = \frac{\omega^2}{c_o^2}\epsilon_3 - \frac{\omega^2}{u^2}\frac{\epsilon_3}{\epsilon_1}. \tag{20}$$

The solution of (19) is easily obtained by the method of the Hankel transform. The result is

$$H_\phi(\rho, \omega) = \frac{iq_o\bar{k}}{8} H_1^{(1)}(\bar{k}\rho), \tag{21}$$

where

$$\bar{k} = \pm k. \tag{22}$$

The sign in (22) is chosen in order to satisfy the radiation condition which requires an outward flow of power at large distances from the source.

In the absence of an external magnetic field, *i.e.*, when the plasma is isotropic, (9) with (20) reduces to

$$k^2 = \frac{\omega^2}{c_o^2} - \frac{\omega_p^2}{c_o^2} - \frac{\omega^2}{u^2}. \tag{23}$$

Since the velocity u of the point charge is always less than the velocity c_o of the electromagnetic waves in free space, it follows from (23) that k is purely imaginary. Hence (21) shows that $H_\phi(\rho, \omega)$ rapidly decays from the source and there is no radiation.

When the external magnetic field is infinite, that is, in a uniaxially anisotropic plasma, it is found from (9) and (20) that

$$k^2 = \left(\frac{1}{c_o^2} - \frac{1}{u^2}\right)(\omega^2 - \omega_p^2). \tag{24}$$

From (24) it is evident that k is real for $\omega < \omega_p$ and is imaginary for $\omega > \omega_p$. Therefore, it follows from (21) that there is radiation only for $\omega < \omega_p$. The power radiated per unit path length in the frequency interval between ω and $\omega + d\omega$ is given by

$$I(\omega)d\omega = 2\pi\rho S_\rho d\omega = \text{Re}\left[\pi\rho E_y(r, \omega)H_\phi^*(r, \omega)\right]d\omega, \tag{25}$$

where S_ρ is the ρ component of the Poynting vector. The expressions for $E_y(r, \omega)$, and $H_\phi(r, \omega)$, needed for the evaluation of $I(\omega)$ in (25) are obtained with the use of (10), (14) and (21). The Hankel function appearing in these expressions is replaced by the leading term of its asymptotic expansion for large k. When only the leading terms are retained, the result is

$$H_\phi(r, \omega) = \frac{iq_o\bar{k}}{8}\sqrt{\frac{2}{\pi k\rho}}\, e^{i(\bar{k}\rho \mp 3\pi/4)} e^{i(\omega/u)y}$$

$$E_y(r, \omega) = \frac{iq_o k^2}{8\omega\epsilon_o\epsilon_3}\sqrt{\frac{2}{\pi k\rho}}\, e^{i(\bar{k}\rho \mp 3\pi/4)} e^{i(\omega/u)y}. \tag{26}$$

The substitution of (26) in (25) gives

$$I(\omega) = \frac{q_o^2 k\bar{k}}{32\omega\epsilon_o\epsilon_3}. \tag{27}$$

Since in the propagation range $\omega < \omega_p$, ϵ_3 is negative, (27) will be positive only when $\bar{k} = -k$. In that case, at large distances from the source, the net flow of power will be directed outward from the source, thus satisfying the radiation condition. Therefore, on using (24) and (9), (27) reduces to

$$I(\omega) = \frac{q_o^2\omega_p}{32\epsilon_o u^2}\left(1 - \frac{u^2}{c_o^2}\right)\frac{\omega}{\omega_p}. \tag{28}$$

$I(\omega)$ given in (28) is the total energy radiated per unit frequency interval per unit path length. In Fig. 2 the frequency spectrum $I(\omega)$ is plotted as a function of ω. The total energy radiated per unit length of path is given by

$$W = 2\int_0^{\omega_p} I(\omega)d\omega = \frac{q_o^2}{64\epsilon_o}\left(\frac{1}{u^2} - \frac{1}{c_o^2}\right)\omega_p^2. \tag{29}$$

The y component of the Poynting vector S_y given by

$$S_y = -\tfrac{1}{2}\,\text{Re}\, E_\rho(r, \omega)H_\phi^*(r, \omega) \tag{30}$$

is obtained with the use of (13) and (21) as

$$S_y = \frac{-q_o^2\omega_p}{64\pi\epsilon_o u^2\rho}\left[\left(1 - \frac{u^2}{c_o^2}\right)\left(1 - \frac{\omega^2}{\omega_p^2}\right)\right]^{1/2}. \tag{31}$$

The angle θ between the direction of motion of the charge and the Poynting vector, or the direction of the

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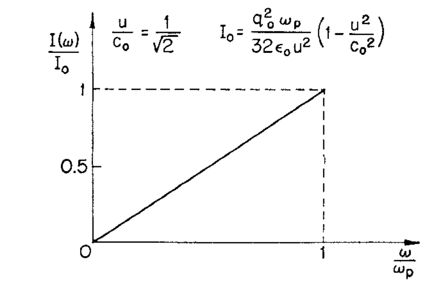

Fig. 2—Frequency spectrum $B_o=\infty$.

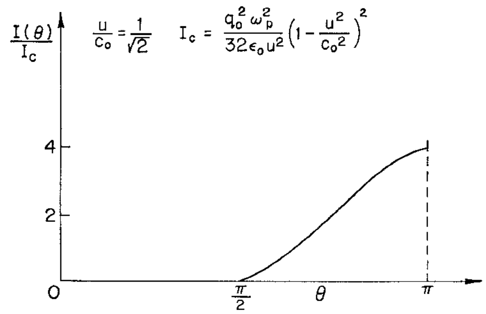

Fig. 3—Angular spectrum $B_o=\infty$.

Cerenkov ray is given by

$$\tan\theta=\frac{S_\rho}{S_y}=-\left(1-\frac{u^2}{c_o^2}\right)^{1/2}\left(\frac{\omega_p^2}{\omega^2}-1\right)^{-1/2}. \quad (32)$$

In the propagation range $\omega<\omega_p$, the right-hand side is seen to be negative and hence shows that the Cerenkov ray makes an obtuse angle with the direction of motion of charge. It is instructive to examine the angular distribution of the radiated energy. The radiation is obviously circularly symmetrical about the direction of travel of the charge. The angular spectrum $I(\theta)$ may be defined by the relation $I(\theta)\sin\theta|d\theta|=I(\omega)d\omega$. Hence, from (28) and (32), it follows that

$$I(\theta)=\frac{q_o^2\omega_p^2}{32\epsilon_o u^2}\left(1-\frac{u^2}{c_o^2}\right)^2\frac{\cos\theta}{\left[1-\frac{u^2}{c_o^2}\cos^2\theta\right]^2}. \quad (33)$$

The angular spectrum is plotted in Fig. 3. From an examination of (32), it is evident that the lowest frequency is directed behind the source, whereas the frequencies close to ω_p are radiated at right angles to the direction of the source. The frequency in the direction of maximum radiation will depend on the ratio u/c_o. At a given time, the entire radiation is contained in the half space behind the source.

Solution of the Coupled Wave-Equations (15) and (16)

It is proposed to solve the coupled wave-equations by the method of the Hankel transform. For that purpose, let the following Hankel transforms of order 1 be defined

$$\overline{H}_\phi(\zeta,\omega)=\int_0^\infty H_\phi(\rho,\omega)J_1(\zeta\rho)\rho d\rho \quad (34a)$$

$$H_\phi(\rho,\omega)=\int_0^\infty \overline{H}_\phi(\zeta,\omega)J_1(\zeta\rho)\zeta d\zeta \quad (34b)$$

$$\overline{E}_\phi(\zeta,\omega)=\int_0^\infty E_\phi(\rho,\omega)J_1(\zeta\rho)\rho d\rho \quad (35a)$$

$$E_\phi(\rho,\omega)=\int_0^\infty \overline{E}_\phi(\zeta,\omega)J_1(\zeta\rho)\zeta d\zeta, \quad (35b)$$

where J_1 is the first-order Bessel function. It is convenient to introduce the following shorthand notations:

$$k_1^2=\frac{\omega^2}{c_o^2}\epsilon_3-\frac{\omega^2}{u^2}\frac{\epsilon_3}{\epsilon_1} \quad (36)$$

$$k_2^2=\frac{\omega^2}{c_o^2}\frac{\epsilon}{\epsilon_1}-\frac{\omega^2}{u^2}. \quad (37)$$

When Hankel transformations (34a) and (35a) are applied to (15) and (16) and the resulting algebraic equations for $\overline{E}_\phi(\zeta,\omega)$ and $\overline{H}_\phi(\zeta,\omega)$ are solved, it follows that

$$\overline{H}_\phi(\zeta,\omega)=\frac{q_o\zeta(k_2^2-\zeta^2)}{4\pi\Delta} \quad (38)$$

and

$$\overline{E}_\phi(\zeta,\omega)=\frac{iq_o\zeta\omega^2\mu_o\epsilon_2}{4\pi u\epsilon_1\Delta}, \quad (39)$$

where

$$\Delta=(k_1^2-\zeta^2)(k_2^2-\zeta^2)-\frac{\omega^4\epsilon_2^2\epsilon_3}{c_o^2u^2\epsilon_1^2}$$
$$=(\zeta^2-k_o^2)(\zeta^2-k_e^2). \quad (40)$$

Since $\overline{H}_\phi(\zeta,\omega)$ and $\overline{E}_\phi(\zeta,\omega)$ are odd functions of ζ, it is possible to recast (34b) and (35b) in the following form:

$$H_\phi(\rho,\omega)=\frac{1}{2}\int_{-\infty}^\infty \overline{H}_\phi(\zeta,\omega)H_1^{(1)}(\zeta\rho)\zeta d\zeta \quad (41)$$

$$\overline{E}_\phi(\rho,\omega)=\frac{1}{2}\int_{-\infty}^\infty \overline{E}_\phi(\zeta,\omega)H_1^{(1)}(\zeta\rho)\zeta d\zeta. \quad (42)$$

With the help of (38)–(42), it follows that

$$H_\phi(\rho,\omega)=\frac{q_o}{8\pi}\int_{-\infty}^\infty\frac{\zeta(k_2^2-\zeta^2)}{(\zeta^2-k_o^2)(\zeta^2-k_e^2)}H_1^{(1)}(\zeta\rho)\zeta d\zeta \quad (43)$$

$$E_\phi(\rho,\omega)=\frac{iq_o\omega^2\mu_0\epsilon_2}{8\pi u\epsilon_1}\int_{-\infty}^\infty\frac{\zeta}{(\zeta^2-k_o^2)(\zeta^2-k_e^2)}H_1^{(1)}(\zeta\rho)\zeta d\zeta. \quad (44)$$

The integral (43) may be evaluated to give the following result:

$$H_\phi(\rho,\omega)=H_{\phi o}(\rho,\omega)+H_{\phi e}(\rho,\omega), \quad (45)$$

where

$$H_{\phi o}(\rho, \omega) = -\frac{iq_o\bar{k}_o}{8}\frac{k_o^2 - k_2^2}{k_o^2 - k_e^2}H_1^{(1)}(\bar{k}_o\rho) \quad (46a)$$

$$H_{\phi e}(\rho, \omega) = \frac{iq_o\bar{k}_e}{8}\frac{k_e^2 - k_2^2}{k_o^2 - k_e^2}H_1^{(1)}(\bar{k}_e\rho). \quad (46b)$$

In (46), $\bar{k}_o$ and $\bar{k}_e$ are given by

$$\bar{k}_o = \pm k_o, \qquad \bar{k}_e = \pm k_e. \quad (47)$$

The sign in (47) is chosen so as to obtain a net outward flow of power at large distances from the source. The result of the evaluation of the integral (44) can be conveniently given as follows:

$$E_\phi(\rho, \omega) = E_{\phi o}(\rho, \omega) + E_{\phi e}(\rho, \omega), \quad (48)$$

$$E_{\phi o}(\rho, \omega) = Z_o H_{\phi o}(\rho, \omega), \quad E_{\phi e}(\rho, \omega) = Z_e H_{\phi e}(\rho, \omega), \quad (49)$$

$$Z_0 = \frac{-i\omega^2\mu_o\epsilon_2}{u\epsilon_1(k_o^2 - k_2^2)}, \qquad Z_e = \frac{-i\omega^2\mu_o\epsilon_2}{u\epsilon_1(k_e^2 - k_2^2)}\cdot \quad (50)$$

It is seen from (45) and (48) that, in general, there are two possible modes which are denoted by subscripts o and e. In the preceding section it was found that in the case of an infinite external magnetic field there is only one mode with wave number given by (24). Of the two modes which are possible in the presence of a finite external magnetic field, that which in the limiting case of infinite external magnetic field has the same wave number as given by (24) is designated with a subscript o (which stands for the ordinary mode). The other mode which arises only when the external magnetic field is finite is called the extraordinary mode and is denoted with a subscript e.

It is evident that the ordinary and extraordinary modes propagate only when k_o and k_e are positive and real. The modes are nonradiating when the corresponding k has an imaginary part. The sign of the roots of (40), therefore, determines whether the corresponding mode is propagating or not and is obviously dependent of the parameters ω_p, ω_c and u. Since the determination of the signs of the roots of (40) turns out to be difficult in the general case, the dispersion curves $\omega - k_o$ and $\omega - k_e$ are examined for some special values of the parameters. The nature of the radiated fields is then examined for these special cases.

Dispersion Relations

In order to find out the frequency spectrum of the radiated energy, the frequency ranges for which k_o and k_e are positive should be determined. Before proceeding to examine the dependence of k_o and k_e on ω, it is desired to point out that, among others, Allis[13] has discussed the characteristics of plane wave propagation in an anisotropic dielectric using wave-normal surfaces which by a transformation[14] can be used to investigate the Cerenkov radiation spectrum. For explicit determination of the frequency ranges of propagation it is simple and convenient to determine in a straightforward manner the functional dependence of k_o and k_e explicitly in terms of ω. It follows from (36), (37) and (40) that

$$k^2_{o,e} = \frac{\omega^2}{2u^2}\left[-b \pm \sqrt{b^2 - 4c}\right], \quad (51)$$

where

$$b = \left[\frac{\epsilon_1 + \epsilon_3}{\epsilon_1} - \frac{u^2}{c_o^2\epsilon_1}(\epsilon_1^2 - \epsilon_2^2 + \epsilon_1\epsilon_3)\right] \quad (52)$$

and

$$c = \left[\frac{\epsilon_3}{\epsilon_1} - 2\epsilon_3\left(\frac{u}{c_o}\right)^2 + \left(\frac{u}{c_o}\right)^4\frac{\epsilon_3(\epsilon_1^2 - \epsilon_2^2)}{\epsilon_1}\right]. \quad (53)$$

When the external magnetic field is infinite, it is seen from (9) that $\epsilon_1 = 1$ and $\epsilon_2 = 0$. For this special case, it is found from (52) and (53) that

$$b = (1 + \epsilon_3)\left(1 - \frac{u^2}{c_o^2}\right), \qquad c = \epsilon_3\left(1 - \frac{u^2}{c_o^2}\right)^2. \quad (54)$$

The substitution of (54) in (51) yields

$$k^2_{o,e} = \left(\frac{1}{u^2} - \frac{1}{c_o^2}\right)(\omega^2 - \omega_p^2), \qquad \left(\frac{1}{c_o^2} - \frac{1}{u^2}\right)\omega^2, \quad (55)$$

where the first term corresponds to the upper sign in (51) and the second term corresponds to the lower sign. Comparison of (55) with (24) shows that the upper and lower signs in (51) correspond, respectively, to the ordinary and the extraordinary modes.

It is convenient at this stage to specialize and let $u/c_o = 1/\sqrt{2}$. With the help of (9), therefore, (51) and (52) become

$$b = \frac{2\Omega^4 - 2\Omega^2R^2 + R^2 - 2}{2\Omega^2(\Omega^2 - R^2 - 1)} \quad (56)$$

and

$$4c = \frac{(\Omega^2 - 1)(\Omega^2 + R\Omega + 1)(\Omega^2 - R\Omega + 1)}{\Omega^4(\Omega^2 - R^2 - 1)}, \quad (57)$$

where

$$\Omega = \frac{\omega}{\omega_p} \quad \text{and} \quad R = \frac{\omega_c}{\omega_p}\cdot \quad (58)$$

The general behavior of b as a function of Ω is different depending on whether $R < \sqrt{2}$ or $R > \sqrt{2}$ and these are

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depicted in Fig. 4. When $R<\sqrt{2}$, b has only one zero at $\Omega=\Omega_2$ given by

$$\Omega_2^2 = \frac{R^2 + [(R^2-1)^2+3]^{1/2}}{2}. \tag{59}$$

For $R>\sqrt{2}$, in addition to the zero at $\Omega=\Omega_2$, b has another zero at $\Omega=\Omega_1$, where

$$\Omega_1^2 = \frac{R^2 - [(R^2-1)^2+3]^{1/2}}{2}. \tag{60}$$

At $\Omega=\Omega_6=\sqrt{1+R^2}$, b has a singularity. As Ω tends to infinity, b asymptotically approaches unity.

An examination of (57) reveals that the general behavior of $4c$ as a function of Ω, is different depending on whether $R>2$ or $R<2$, (Fig. 5). For $R<2$, $4c$ has a zero for $\Omega=1$ and a singularity at $\Omega=\Omega_6$. When Ω tends to infinity, $4c$ asymptotically approaches unity. However, for $R>2$, $4c$ has two additional zeros at Ω_3 and Ω_4 given by

$$\Omega_3 = \frac{R-\sqrt{R^2-4}}{2} \tag{61a}$$

$$\Omega_4 = \frac{R+\sqrt{R^2-4}}{2}. \tag{61b}$$

It is seen from (51) that the behavior of b^2-4c as a function of Ω is necessary for the determination of the range of frequencies for which k_o and k_e are positive and real. It can be shown with the help of (56) and (57) that

$$b^2 - 4c = \frac{2R^2\left[\Omega^2 + \frac{R^2}{8} - 1\right]}{\Omega^4[\Omega^2 - R^2 - 1]^2}. \tag{62}$$

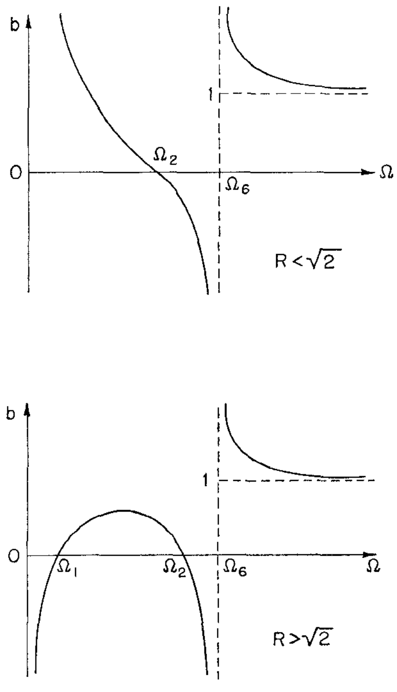

Fig. 4—b as the function of Ω.

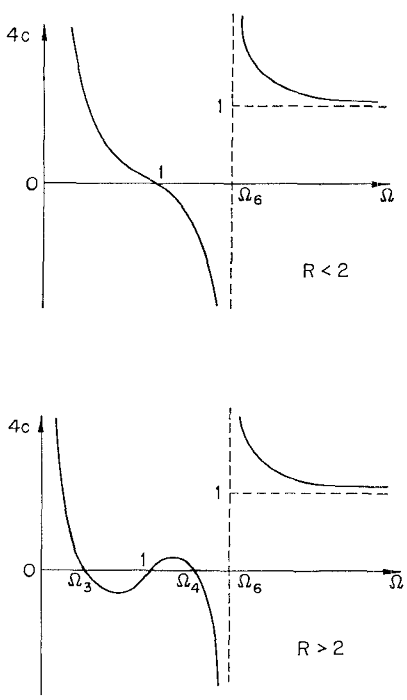

Fig. 5—The general behavior of $4c$ as a function of Ω.

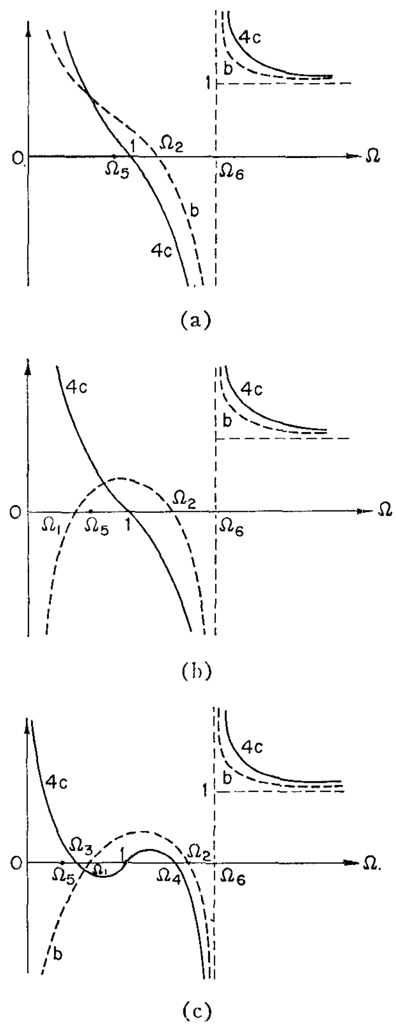

Fig. 6—Behavior of band $4c$ as a function of Ω. (a) Case 1): $0<R<\sqrt{2}$ (b) Case 2): $\sqrt{2}<R<2$. (c) Case 3): $2<R<\infty$.

From (62) it is obvious that $b^2-4c \lesseqgtr 0$ depending on whether $\Omega \lesseqgtr \Omega_5$ where

$$\Omega_5^2 = 1 - \frac{R^2}{8} \cdot \qquad (63)$$

It is evident that the general behavior of k_o and k_e as a function of Ω will be different for the following three different ranges of R, namely 1) $0<R<\sqrt{2}$ 2) $\sqrt{2}<R<2$ and 3) $2<R<\infty$. In Fig. 6, b and $4c$ are plotted as a function of Ω for the three different ranges of R. In particular, for each case, the relative values of 1, $\Omega_1-\Omega_6$ are made clear.

Case 1)

When $\Omega_6<\Omega<\infty$, $b^2-4c>0$; also $4c>0$ resulting in $|\sqrt{b^2-4c}|<|b|$ and, hence, the signs of k_o^2 and k_e^2 are determined by the sign of $-b$. In that frequency range $b>0$; hence k_o^2 and k_e^2 are both negative and neither of the modes propagates. For the same reason both the modes do not propagate in the frequency range $\Omega_5<\Omega<1$. When $0<\Omega<\Omega_5$, $b^2-4c<0$ and, hence, k_o and k_e will have an imaginary part, as a consequence of which, both the modes are exponentially damped and there is no propagation. For $1<\Omega<\Omega_6$, $4c$ is negative so that $|\sqrt{b^2-4c}|>|b|$. Hence, the upper and the lower signs in (51) yield, respectively, positive and negative values: the ordinary mode, which corresponds to the upper sign in (51), propagates in the range $1<\Omega<\Omega_6$.

Case 2)

It follows from arguments such as those given above that neither of the two modes propagates in the ranges $0<\Omega<1$ and $\Omega_6<\Omega<\infty$ and, as before, the ordinary mode propagates in the range $1<\Omega<\Omega_6$.

Case 3)

As in the two previous cases, it can be argued that in the frequency ranges $0<\Omega<\Omega_5$, $1<\Omega<\Omega_4$ and $\Omega_6<\Omega<\infty$, there is no propagation. Only the ordinary mode propagates in the two frequency ranges $\Omega_3<\Omega<1$ and $\Omega_4<\Omega<\Omega_6$.

In the frequency range $\Omega_5<\Omega<\Omega_3$, $b^2-4c>0$; also since $4c>0$, $|\sqrt{b^2-4c}|<|b|$. Therefore, the sign of k_o^2 and k_e^2 is the same as that of $-b$. Since $b<0$ in that range of frequencies, both k_o and k_e will be positive, and hence, both the ordinary and extraordinary modes propagate in that frequency range.

From the foregoing discussions, it may be concluded that for $R<2$, only the ordinary mode propagates in the frequency range $1<\Omega<\Omega_6$. But for $R>2$ both modes propagate, the ordinary mode in the two frequency ranges $\Omega_5<\Omega<1$ and $\Omega_4<\Omega<\Omega_6$ and the extraordinary mode in the frequency range $\Omega_5<\Omega<\Omega_3$.

In what follows, two special values are chosen for R, namely $R=1$ and $R=2\sqrt{2}$, which lie respectively in the two different ranges $R<2$ and $R>2$. Note that $R=2\sqrt{2}$ gives the widest frequency spectrum for the extraordinary mode. In Fig. 7 the propagation ranges for the two modes are shown pictorially for the two special values of R.

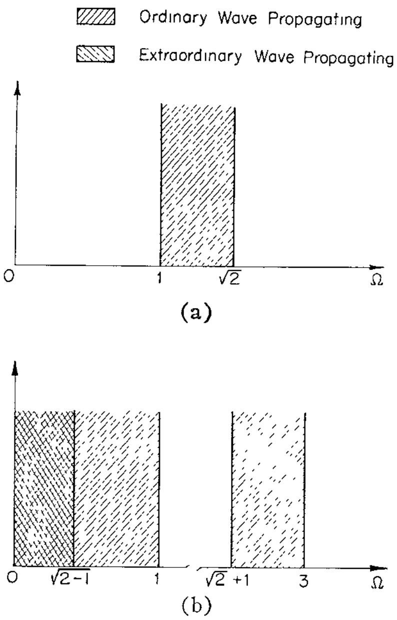

Fig. 7—Propagation ranges for the ordinary and extraordinary modes. (a) Case 1): $R=1$. (b) Case 2): $R=2\sqrt{2}$.

Frequency Spectrum of the Radiated Energy

It is possible to show that the ordinary and the extraordinary modes are orthogonal and hence, the total power radiated can be obtained as the sum of the powers in each of the two modes separately. The powers radiated in the ordinary and extraordinary modes per unit path length in the frequency interval between ω and $\omega+d\omega$ are given, respectively, by

$$I_o(\omega)d\omega = 2\pi\rho S_{\rho o}d\omega \qquad I_e(\omega)d\omega = 2\pi\rho S_{\rho e}d\omega, \qquad (64)$$

where $S_{\rho o}$ and $S_{\rho e}$ are the ρ components of the Poynting vector for the two modes. They are obtained from the following relations:

$$S_{\rho o} = \tfrac{1}{2} \operatorname{Re} \left[E_{yo}(r,\omega)H_{\phi o}^*(r,\omega) - E_{\phi o}(r,\omega)H_{yo}^*(r,\omega)\right] \quad (65a)$$

$$S_{\rho e} = \tfrac{1}{2} \operatorname{Re} \left[E_{ye}(r,\omega)H_{\phi e}^*(r,\omega) - E_{\phi e}(r,\omega)H_{ye}^*(r,\omega)\right]. \quad (65b)$$

The expressions for $H_\phi(r,\omega)$, $E_\phi(r,\omega)$, $H_y(r,\omega)$ and $E_y(r,\omega)$ needed for the evaluation of S_ρ in (65) are obtained with the use of (10), (12), (14), (46) and (49). The Hankel function appearing in these expressions is replaced by the leading term of its asymptotic expansion for large k. The result, when only the leading terms are retained, is as follows:

Ordinary mode

$$H_{\phi o}(r,\omega) = -\frac{iq_o\bar{k}_o}{8}\frac{k_o^2-k_2^2}{k_o^2-k_e^2}\sqrt{\frac{2}{\pi k_o\rho}} \cdot e^{i(\bar{k}_o\rho\mp 3\pi/4)}e^{i(\omega/u)y}$$

$$E_{\phi o}(r,\omega) = -\frac{q_o\bar{k}_o}{8}\frac{\omega^2\mu_o\epsilon_2}{u\epsilon_1}\frac{1}{k_o^2-k_e^2}\sqrt{\frac{2}{\pi k_o\rho}} \cdot e^{i(\bar{k}_o\rho\mp 3\pi/4)}e^{i(\omega/u)y}$$

$$H_{yo}(r,\omega) = -\frac{\bar{k}_o}{\omega\mu_o}E_{\phi o}(r,\omega)$$

$$E_{yo}(r,\omega) = \frac{\bar{k}_o}{\omega\epsilon_o\epsilon_3}H_{\phi o}(r,\omega). \tag{66}$$

Extraordinary mode

$$H_{\phi e}(r,\omega) = \frac{iq_o\bar{k}_e}{8}\frac{k_e^2-k_2^2}{k_o^2-k_e^2}\sqrt{\frac{2}{\pi k_e\rho}} \cdot e^{i(\bar{k}_e\rho\mp 3\pi/4)}e^{i(\omega/u)y}$$

$$E_{\phi e}(r,\omega) = \frac{q_o\bar{k}_e}{8}\frac{\omega^2\mu_o\epsilon_2}{u_1\epsilon_1}\frac{1}{k_o^2-k_e^2}\sqrt{\frac{2}{\pi k_e\rho}} \cdot e^{i(\bar{k}_e\rho\mp 3\pi/4)}e^{i(\omega/u)y}$$

$$H_{ye}(r,\omega) = -\frac{\bar{k}_e}{\omega\mu_o}E_{\phi e}(r,\omega)$$

$$E_{ye}(r,\omega) = \frac{\bar{k}_e}{\omega\epsilon_o\epsilon_3}H_{\phi e}(r,\omega). \tag{67}$$

The substitution of (65)–(67) in (64) yields

$$I_o(\omega) = \frac{q_o^2k_o\bar{k}_o}{32\omega(k_o^2-k_e^2)}\left[\frac{(k_o^2-k_2^2)^2}{\epsilon_o\epsilon_3}+\frac{\omega^4\epsilon_2^2\mu_o}{\epsilon_1^2u^2}\right] \tag{68a}$$

and

$$I_e(\omega) = \frac{q_o^2k_e\bar{k}_e}{32\omega(k_o^2-k_e^2)}\left[\frac{(k_e^2-k_2^2)^2}{\epsilon_o\epsilon_3}+\frac{\omega^4\epsilon_2^2\mu_o}{\epsilon_1^2u^2}\right]. \tag{68b}$$

$I_o(\omega)$ given in (68a) is the total energy radiated per unit frequency interval per unit path length; it is called the frequency spectrum of the ordinary mode. Similarly, $I_e(\omega)$ is the frequency spectrum of the extraordinary mode.

It is desired to examine the frequency spectrum for the particular value of $u/c_o=1/\sqrt{2}$ and for the two cases $R=1$ and $R=\sqrt{2}$.

Case 1): $R=1$

In this case, the extraordinary mode is not excited and, hence, $I_e(\omega)=0$. The ordinary mode propagates in the range $1<\Omega<\sqrt{2}$. For $u/c_o=1/\sqrt{2}$ and $R=1$, (68a) can be simplified to yield the following expression for the frequency spectrum:

$$I_o(\omega) = \pm\frac{q_o^2\omega_p}{32u^2\epsilon_o} \cdot \frac{\Omega\left[3-4(\Omega^2-\frac{1}{2})^2-2\sqrt{8\Omega^2-7}\right]\left[\sqrt{8\Omega^2-7}+1\right]}{(\Omega^2-1)(\Omega^2-2)\sqrt{8\Omega^2-7}}, \tag{69}$$

where the upper and lower signs correspond to the upper and lower signs in (47). In the frequency range $1<\Omega<\sqrt{2}$, $I_o(\omega)$ is positive only if the upper sign is chosen. The choice of upper sign in (69) insures that the net flow of power at large distances from the source is directed outward and thus the fulfillmenent of the radiation condition is assured by the choice of the upper sign in (47). The result is a cylindrical wave with outward traveling phase-front. In Fig. 8 the frequency spectrum is plotted as a function of Ω.

Case 2): $R=2\sqrt{2}$

Both the ordinary and extraordinary modes are excited when $R=2\sqrt{2}$. The ordinary mode propagates in the frequency ranges $0<\Omega<1$ and $\sqrt{2}+1<\Omega<3$ and the extraordinary mode, for $0<\Omega<\sqrt{2}-1$. On setting $u/c_o=1/\sqrt{2}$ and $R=2\sqrt{2}$ in (68a), the resulting expression, after some simplification, reduces to the following:

$$I_o(\omega) = \pm\frac{q_o^2\omega_p}{128\epsilon_ou^2}\frac{\Omega^4-8\Omega^2+4\Omega+3}{(9-\Omega^2)(\Omega-1)}, \tag{70}$$

where, as before, the upper and lower signs correspond to upper and lower signs in (47). In the frequency range $0<\Omega<1$, $I_o(\omega)$ is positive only if the lower sign is chosen. The choice of the lower sign in (70) leads to the fulfillment of the radiation condition. This choice of sign in (47) shows that the ordinary mode in the frequency range $0<\Omega<1$ has an inward traveling phase-front; it may be called a "backward wave." However, in the frequency range $\sqrt{2}+1<\Omega<3$, the upper sign in (70), and hence the upper sign in (47), should be chosen in order that the radiation condition be satisfied. In Fig. 9(a) the frequency spectrum of the ordinary wave is plotted as a function of Ω.

Similarly, the substitution of $u/c_o=1/\sqrt{2}$ and $R=2\sqrt{2}$ in (68b) will yield, after some simplification, the following expression for the frequency spectrum of the extraordinary wave:

$$I_e(\omega) = \pm\frac{q_o^2\omega_p}{128\epsilon_ou^2}\frac{\Omega^4-8\Omega^2-4\Omega+3}{(9-\Omega^2)(\Omega+1)}. \tag{71}$$

In the range $0<\Omega<\sqrt{2}-1$, $I_e(\omega)$ in (71) is positive only when the upper sign is chosen. Therefore, for the extraordinary wave, the upper sign should be chosen in (47) in order that the radiation condition be fulfilled. The frequency spectrum of the extraordinary wave is shown in Fig. 9(b).

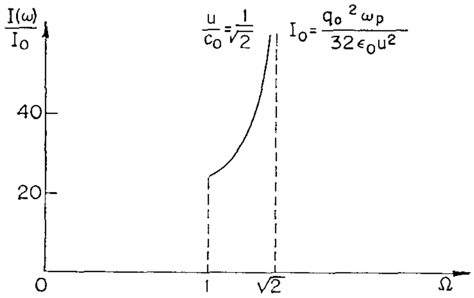

Fig. 8—Frequency spectrum of ordinary wave $R=1$.

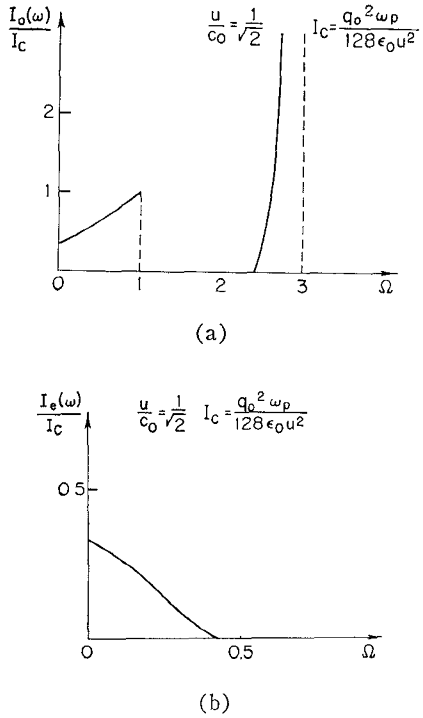

Fig. 9—(a) Frequency spectrum of ordinary wave $R=2\sqrt{2}$. (b) Frequency spectrum of extraordinary wave $R=2\sqrt{2}$.

An examination of Figs. 9(a) and (b) shows that a uniformly moving charge radiates *both* modes in the frequency range $0<\Omega<\sqrt{2}-1$.

Angular Spectrum of the Radiated Energy

A point charge uniformly moving in an anisotropic medium in the direction of an external magnetic field, in general, radiates two modes. In the preceding section, the frequency spectrum of the emitted radiation was studied. From (46) it is clear that this radiation is in the form of a cylindrical wave and different frequency components, evidently, will radiate in different directions with respect to the direction of motion of the charge. Also, the same frequency components in different modes may radiate in different directions. It is therefore of interest to study the angular spectrum of the radiated energy. The determination of the angular spectrum involves a great deal of algebraic manipulation and numerical calculation. For the sake of brevity, details are omitted and only the essential steps are indicated. The y components of the Poynting vector S_{yo} and S_{ye} for the ordinary and extraordinary modes are first obtained from the following relations:

$$S_{yo}=\tfrac{1}{2}\,\mathrm{Re}\left[E_{\phi o}(r,\omega)H_{\rho o}^*(r,\omega)-E_{\rho o}(r,\omega)H_{\phi o}^*(r,\omega)\right] \quad (72a)$$

and

$$S_{ye}=\tfrac{1}{2}\,\mathrm{Re}\left[E_{\phi e}(r,\omega)H_{\rho e}^*(r,\omega)-E_{\rho e}(r,\omega)H_{\phi e}^*(r,\omega)\right]. \quad (72b)$$

The angle θ which the Cerenkov ray makes with the direction of motion of the charge (see Fig. 1) is given by the following expressions:

$$\tan\theta = \frac{S_{\rho o}}{S_{yo}} \quad (73a)$$

$$\tan\theta = \frac{S_{\rho e}}{S_{ye}} \quad (73b)$$

where (73a) and (73b), respectively, refer to the ordinary and extraordinary modes. The angular spectrum $I_o(\theta)$ and $I_e(\theta)$ for the ordinary and the extraordinary modes are obviously given by the relations

$$I_o(\theta)\sin\theta\,|d\theta| = I_o(\omega)d\omega \quad (74a)$$

and

$$I_e(\theta)\sin\theta\,|d\theta| = I_e(\omega)d\omega. \quad (74b)$$

With the help of (65), (68), (72)–(74), $I_o(\theta)$ and $I_e(\theta)$ can be computed. The results are plotted in Figs. 10, 11 and 12. For $R=1$, only the ordinary wave is excited and its angular spectrum is plotted in Fig. 10. It is seen that the entire radiation is confined to the half space trailing the source. The low-frequency end of the spectrum is radiated at right angles to the direction of motion of the source and the high-frequency end of the spectrum is radiated directly behind the source.

For the case $R=2\sqrt{2}$ both the ordinary and the extraordinary waves are excited. The angular spectrum of the ordinary wave for the frequency components lying in the range $0<\Omega<1$ are plotted in Fig. 11(a) and that for $\sqrt{2}+1<\Omega<3$ in Fig. 11(b). It is seen that the entire radiation corresponding to the frequencies lying in the range of $0<\Omega<1$ is confined to the half space trailing the source. The frequency components near zero are radiated directly behind the source, whereas the frequency components near the plasma frequency are radiated at right angles to the direction of motion of the source.

The ordinary wave of frequency components lying in the range $\sqrt{2}+1<\Omega<3$ are partly radiated in the forward and partly in the backward directions. The fre-

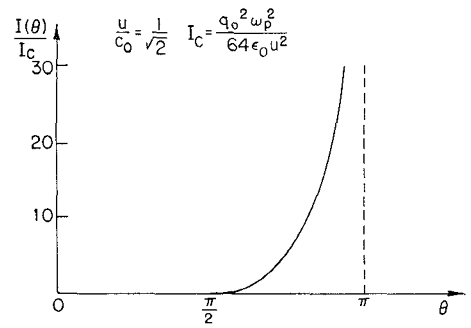

Fig. 10—Angular spectrum of ordinary wave $R=1$.

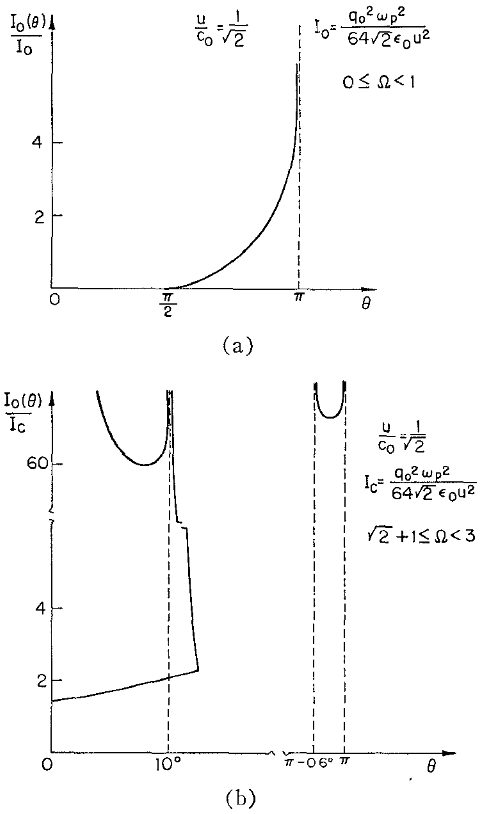

Fig. 11—(a) Angular spectrum of ordinary wave $R=2\sqrt{2}$. (b) Angular spectrum of ordinary wave $R=2\sqrt{2}$.

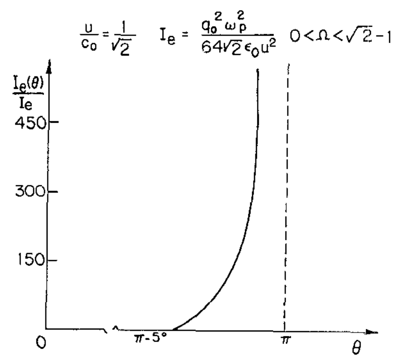

Fig. 12—Angular spectrum of extraordinary wave $R=2\sqrt{2}$.

quency components in the range $\sqrt{2}+1<\Omega<2\sqrt{2}$ are radiated in a narrow cone of an angle about 10° about the direction of motion of source, whereas the frequency components in the range $2\sqrt{2}<\Omega<3$ are confined to a very narrow cone trailing the source. In the forward direction, *i.e.*, for $0<\theta<10°$ approximately, there are two Cerenkov rays corresponding to each direction and these two rays clearly correspond to different frequency components.

The angular spectrum of the extraordinary wave is plotted in Fig. 12. The entire radiation is confined to a narrow cone trailing the source. An examination of Figs. 11 and 12 shows that within a narrow cone of approximately 5°, there are three Cerenkov rays, two of which belong to the ordinary wave, whereas the third belongs to the extraordinary wave. These three Cerenkov rays propagating in the same direction, in general, correspond to different frequency components. Although in the past there have been conjectures on the possibility of the existence of multiple Cerenkov rays traveling in the same direction, it is believed that an explicit evaluation of the multiple Cerenkov rays has been carried out here for the first time.

Acknowledgment

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